Modeling Repairs of Systems with a Bathtub-Shaped Failure Rate Function

Sima Varnosafaderani
School of Mathematics, Statistics and Operations Research
Victoria University of Wellington
New Zealand
sima@msor.vuw.ac.nz

Stefanka Chukova
School of Mathematics, Statistics and Operations Research
Victoria University of Wellington
New Zealand
stefanka@msor.vuw.ac.nz

Abstract

Most of the reliability literature on modeling the effect of repairs on systems assumes the failure rate functions are monotonically increasing. For systems with non-monotonic failure rate functions, most models deal with minimal repairs (which do not affect the working condition of the system) or replacements (which return the working condition to that of a new and identical system). We explore a new approach to model repairs of a system with a non-monotonic failure rate function; in particular, we consider systems with a bathtub-shaped failure rate function. We propose a repair model specified in terms of modifications to the virtual age function of the system, while preserving the usual definitions of the types of repair (minimal, imperfect and perfect repairs) and distinguishing between perfect repair and replacement. In addition, we provide a numerical illustration of the proposed repair model.

1 Introduction

Most engineered systems – defined as an arrangement of components that together perform an identified (and predefined) set of functions – are susceptible to failures, and require some form of rectification in order to return to a functioning condition. Most rectifications have an effect on the probability and number of future failures of the system over a given period of time.

The sequence of numbers of failures of the system in time, i.e. the failure process, is modeled as a stochastic counting process, assuming that there can be at

most one failure in an infinitesimally small interval of time. When the rectification action following each failure is immediate and instantaneous, this process can also be described as the sequence of times to failure (or consecutive system's lifetimes).

A counting process is completely described by its conditional intensity function, and therefore, rectifications are usually defined in terms of their effect on the conditional intensity function of the failure process. The initial conditional intensity function is the failure rate function of the original lifetime (time to first failure of the system), which is often a continuous function of time, and is classified as constant, monotonic increasing or decreasing, or a combination of these. Beyond the first failure, the conditional intensity function is altered in accordance with the rectifications performed following the first and all consequent failures.

Not all rectifications have the same effect on the system, and based on their effect, they are categorized as either replacements or repairs with varying degrees of effectiveness. In some cases, replacements can be viewed as extreme repairs.

In this article, we suggest an approach to model the effect of rectifications (here, repairs) for a system having a non-monotonic (here, bathtub-shaped) failure rate function. We define repairs in terms of their effect on the virtual age of the system.

The article is arranged as follows. In Section 2 we discuss the concepts mentioned above in more detail, and provide a brief review of existing models relevant to our study. In Section 3, we describe the repair model and provide model formulation. In Section 4, we provide a numerical illustration of the proposed model. Finally, in Section 5, we conclude with a discussion of the proposed model and some directions for future research.

2 Background and Definitions

Let $\lambda_c(t)$ denote the conditional intensity function of the failure process denoted by $\{N(t); t \geq 0\}$. Then

$$\lambda_c(t) = \lim_{dt \to 0} \frac{P\{N(t+dt) - N(t) = 1 \mid \mathcal{F}_t\}}{dt} \quad ,$$

where $N(t+dt) - N(t)$ is the number of failures in the interval $(t, t+dt]$, and $\mathcal{F}_t = \{N(s); 0 \leq s < t\}$ is the history of the process before time t. The initial conditional intensity (or baseline intensity), denoted by $\lambda_0(t)$, is the failure rate of the time to first failure, which is

$$\lambda_0(t) = r(t) = \lim_{dt \to 0} \frac{P\{N(t+dt) - N(t) = 1 \mid N(t) = 0\}}{dt} \quad .$$

A failure rate or the corresponding distribution is categorized as: constant failure rate (CFR) when $r(t)$ is constant over t; increasing failure rate (IFR) when $r(t)$ is increasing in t; decreasing failure rate (DFR) when $r(t)$ is decreasing in t; or some combination of these. For instance, the bathtub-shaped failure rate (BFR) function which is initially decreasing, then constant and finally increasing:

$$r(t) = \begin{cases} r_1(t) : r_1'(t) < 0 \ , & t \leq a_1 \\ r_2(t) : r_2'(t) = 0 \ , & a_1 < t \leq a_2 \\ r_3(t) : r_3'(t) > 0 \ , & t > a_2 \ , \end{cases} \tag{1}$$

where a_1 and a_2 are the change points (points at which the the derivative of the failure rate function changes sign) of the BFR function. The BFR function is a generalization of the above categories; setting $a_1 = a_2 = 0$ ($a_1 = a_2 = \infty$ or $a_1 = 0$ and $a_2 = \infty$), it becomes an increasing (decreasing or constant) function. Setting $a_1 = a_2 = a$, we get a U-shaped failure rate (UFR) function (Lai and Xie 2006); see Figure 1.

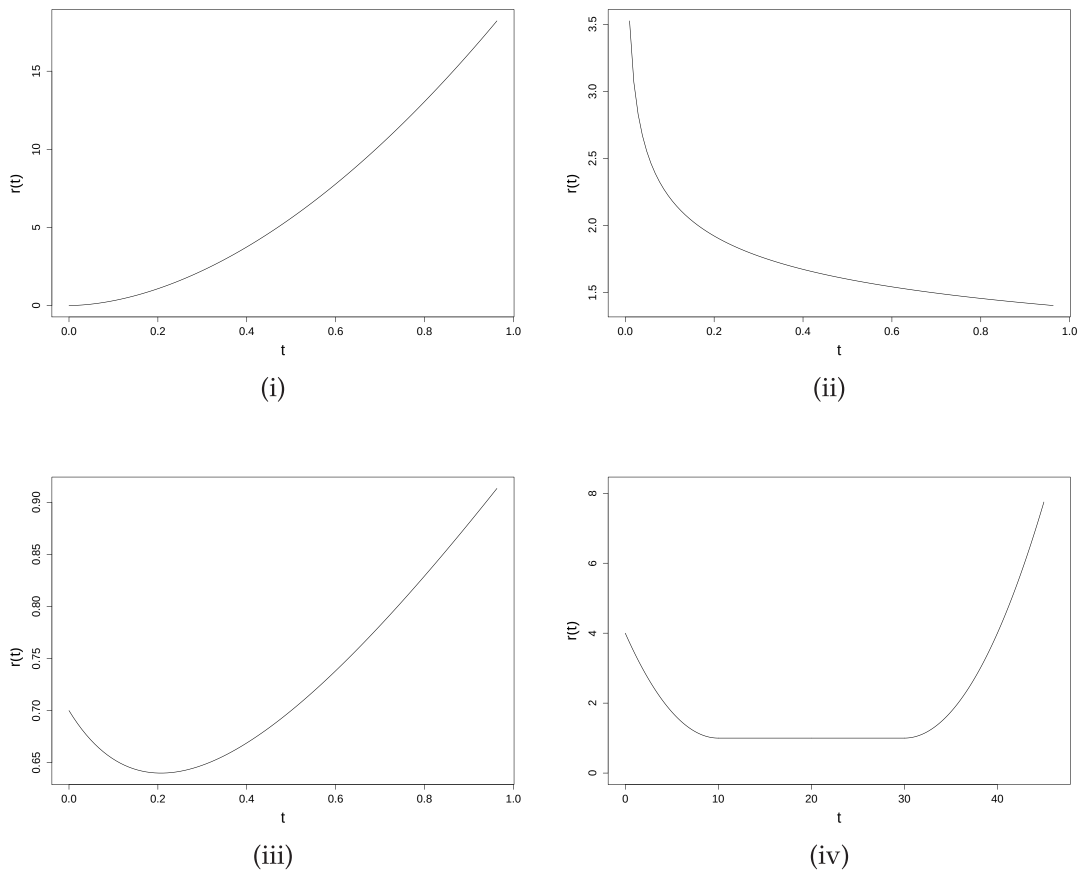

Figure 1: Failure rate functions: (i) IFR; (ii) DFR; (iii) UFR; (iv) BFR.

The quantity $r(t)\,dt$ is the approximate probability that the system will fail for the first time in $(t, t + dt]$. The distribution of the time to first failure, denoted by T_1, is defined by the failure rate $r(t)$, but subsequent failure times are affected by the type of rectification performed after a failure.

Rectifications are broadly classified into repair and replacement. With replacements, the system is replaced by a new, identical system upon failure. The condition of the system following a replacement is therefore identical to a new system. In this case, the failure process is modeled as a renewal process with conditional intensity function $\lambda_c(t) = r(t - T_{N(t)})$ (where $T_{N(t)}$ is the time of the last replacement (perfect repair)), and the expected number of replacements is given by the renewal function (Hokstad 1997; Hunter 1974; Aven and Jensen 1998). Replacements apply when the system is beyond repair (or non-repairable) or when replacing the system is more feasible than repairing it.

Repairs are characterized by their effectiveness, which is often referred to as the *degree of repair* – the degree to which the functioning condition of the system

is restored following the repair. Based on this degree, repairs are typically categorized as minimal, imperfect or perfect repair.

Minimal repairs have no effect on the failure rate function; i.e., the system immediately before and after the failure is in the same functioning condition. Therefore, when all repairs are minimal, the failure process is a non-stationary Poisson process with conditional intensity function $\lambda_c(t) = r(t)$. The expected number of failures over an interval $[0, t)$ is then given by $E[N(t)] = \int_0^t \lambda_c(s)\, ds$.

Perfect repairs, for IFR functions, leave the repaired system in an as-good-as-new working condition, which implies that a perfect repair is equivalent to a replacement and can be modeled as one. This definition works for systems having an IFR function, since the system at the start of its lifetime has the lowest failure rate, i.e., it is at its best. However, if a system has a failure rate function that is initially decreasing (e.g. BFR), this definition does not hold, because a repair that leaves the system in an as-good-as-new condition is actually worsening the system, since the failure rate of the system at the start of its lifetime is higher than its failure rate when it is working at its best. Therefore, we distinguish between replacement and perfect repair, and describe a perfect repair as the best form of repair (not taking into account improvements/upgrades). In other words, a perfect repair is one that restores the functioning condition of the system to its condition when it is performing at its best. This point of ideal performance is at the start of the system's lifetime for a system with an IFR function, but not for a system having an initially decreasing failure rate function.

Imperfect repair, sometimes referred to as general repair, is any repair that leaves the system in a functioning condition that is between the functioning conditions following minimal and perfect repairs. For systems with a IFR function, the definition of an imperfect repair has included the extremes minimal repair and replacement (aka perfect repair). Here, imperfect repair includes as its extremes minimal and perfect repairs, but not replacements. Therefore, we distinguish between repair and replacement.

In most settings, the degree of repair is a variable with range $[0, 1]$, where a degree of zero corresponds to a minimal repair, a degree of one corresponds to a *perfect* repair, and a degree between these extremes corresponds to an imperfect repair. Therefore, the higher the degree of repair, the bigger the improvement in the functioning condition of the system.

Here, we do not consider repairs that can worsen the system or upgrades (or improvements).

Many repair models have been suggested for systems having IFR functions. Some common models are the virtual age models discussed in Kijima (1989), Varnosafaderani and Chukova (2012a) and Doyen and Gaudoin (2004); and the intensity reduction models discussed in Lindqvist (1998), Varnosafaderani and Chukova (2012b) and Doyen and Gaudoin (2004). Models of repair in the case of BFR functions assume that rectifications are either minimal or replacements. The virtual age models for IFR functions have been applied to BFR functions; see (Dijoux 2009), but due to the failure rate being initially decreasing, repairs of degree greater than zero actually worsen the product.

In this article, we propose a new approach to modeling imperfect repairs for systems having BFR functions which better suit the definitions of the types of

repair. The effects of repairs are described in terms of modifications to the virtual age function of the system.

3 Modeling the Effect of Repairs

Let T_i denote the time of the ith failure (also repair, since repairs are immediate and instantaneous), and let δ_i denote the degree of the ith repair. Also, let $A(t)$ denote the virtual age of the system at time t.

Based on the virtual age function of the system, we propose the following repair model. The virtual age of the system at time t, is given by

$$A(t) = \begin{cases} t + \sum_{i=1}^{N(t^-)} \delta_i \left[a_1 - A(T_i)\right] \ , & t \leq a_1 \\ t - \sum_{i=N(a_1^+)}^{N(t^-)} \delta_i \left[A(T_i) - a_1\right] \ , & t > a_1 \end{cases} \tag{2}$$

where $A(T_i)$ is the virtual age of the system at the time of its ith failure. Before the first failure, when $N(t^-) = 0$, the virtual age is simply $A(t) = t$; and immediately after a_1 and before the first failure in the useful life period, the virtual age of the system is again $A(t) = t$. See Figure 2 for an illustration of the virtual age function for three failures occurring at times t_1, t_2 and t_3.

With a failure rate function that is initially decreasing, the point of best performance is not the start of the system's lifetime, but the point at which the failure rate function is at its lowest. This point for a BFR function is the first change point a_1.

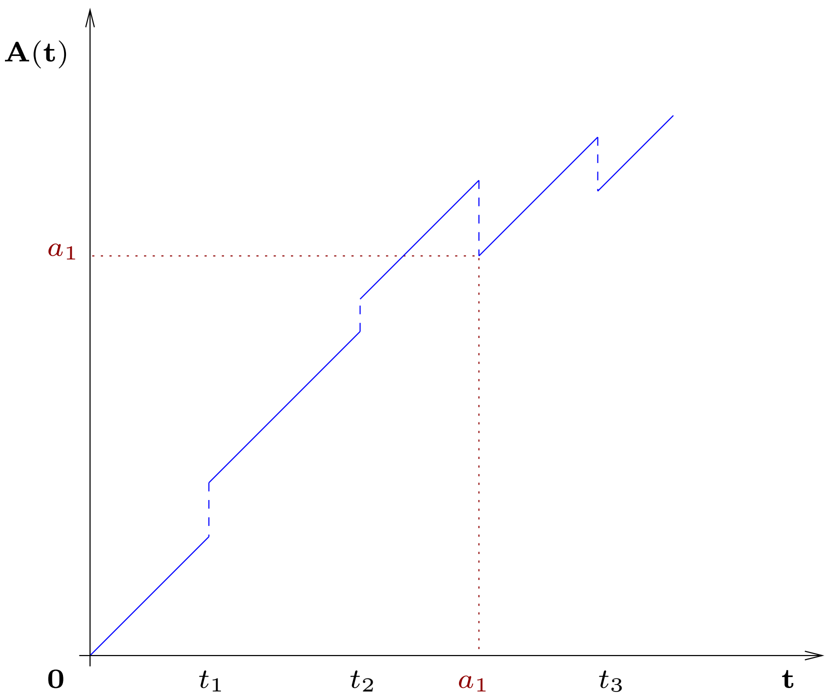

Figure 2: Virtual age function following imperfect repairs of varying degree.

Therefore, according to this model, when the virtual age of the system at the time of the ith failure is less than the first change point a_1, the effect of a repair is

modeled as an increase in the virtual age of the system, such that, a perfect repair results in the virtual age being a_1. At a_1, the virtual age of the system is set to its calendar age, i.e. $A(a_1) = a_1$. This extends the useful life period of the system, which will decrease the probability of future failures. When the age of the system is greater than the first change point a_1, then the effect of a repair is a decrease in the virtual age of the system, such that, a perfect repair results in the virtual age being a_1. The point a_1 is the point of ideal performance, because it is the start of the useful life of the system, and the failure rate of the system at this point is at its lowest.

The conditional intensity function of the failure process is given by

$$\lambda_c(t) = \begin{cases} \lambda_0(t) \ , & t \leq T_1 \\ \lambda_0(A(t)) \ , & t > T_1 \end{cases}$$

where $\lambda_0(.)$ is the baseline intensity (or failure rate) function. See Figure 3 for an illustration of this function following repairs of varying degree.

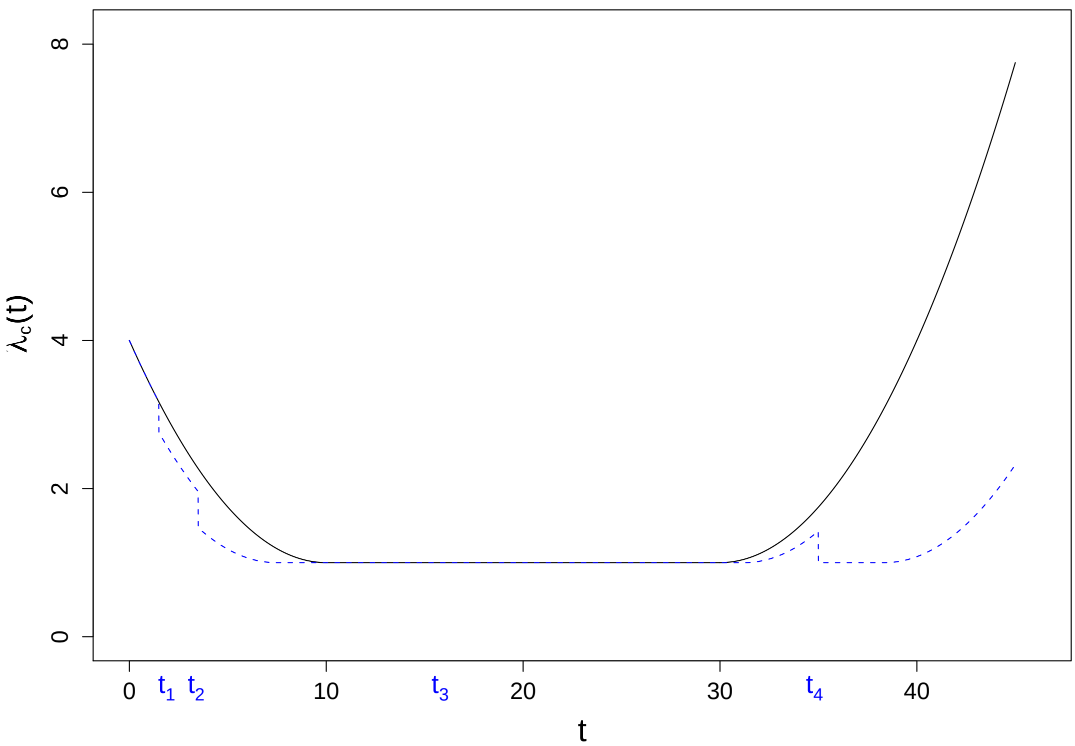

Figure 3: Conditional intensity function following: four imperfect repairs of varying degree (dashed line); minimal repairs (solid line).

The repair model stays true to the definitions of the types of repair. A perfect repair is the best form of repair, and should result in the system performing at its best, which is in this case at a_1. A minimal repair, should by definition leave the system in the same condition that it was prior to failure, and here, the virtual age does not change following a minimal repair. The effect of an imperfect repair should be between those of the minimal and perfect repairs, and effectiveness of the repair should increase with its degree. Here, as the degree of repair increases,

so does the effectiveness of the repair (which is reflected in the decrease in the conditional intensity function of the process).

The assumption for this model is that the useful life period $(a_1, a_2]$ of the system is at least as long as the DFR period $(0, a_1]$, i.e. $a_2 - a_1 \geq a_1$.

4 Numerical Illustration

In this section, we provide a simple example that illustrates the proposed repair model.

The baseline intensity function used in this example is

$$\lambda_0(t) = \begin{cases} \lambda + \alpha_1\ (a_1 - t)^{\beta_1}\ , & t \leq a_1 \\ \lambda\ , & a_1 < t \leq a_2 \\ \lambda + \alpha_2\ (t - a_2)^{\beta_2}\ , & t > a_2\ , \end{cases} \tag{3}$$

where $\lambda > 0$, $\beta_1, \beta_2 > 0$, $\beta_1 \geq \beta_2$, and $\alpha_1, \alpha_2 > 0$. The parameter values are chosen to be $\lambda = 1$, $\alpha_1 = 0.6$, $\alpha_2 = 0.5$, $\beta_1 = 2.5$, and $\beta_2 = 2.8$, and the change points are chosen to be $a_1 = 4$ and $a_2 = 8$.

Since virtual age models for IFR functions have been frequently examined and the effect of repairs in this case is known, we limit our illustration to exploring the effect of repairs based on our virtual age model in the DFR phase. To do so, we select an arbitrary mission time τ, and applying repairs of varying degree in the interval $[0, a_1)$, we compute the expected number of failures in $(0, \tau]$. Here, the mission time is chosen to be $\tau = 10$.

The repairs performed are chosen according to the following strategy: the first repair in the interval $(0, a_1]$ is imperfect, and all other repairs are minimal.

Let T_1 denote the time of the first failure. The density function of T_1 in terms of the baseline intensity function is given by

$$f_1(t) = \lambda_0(t)\ e^{-\int\limits_0^t \lambda_0(s)\ ds}\ . \tag{4}$$

The expected number of failures in the interval $[0, \tau)$, is then derived as follows:

$$E[N(\tau)] = \int\limits_0^{a_1} \left[1 + \int\limits_{t_1}^{a_1} \lambda_0(s + \delta_1(a_1 - t_1))\ ds\right] f_1(t_1)\ dt_1 + \int\limits_{a_1}^{\tau} \lambda_0(s)\ ds\ ,$$

where δ_1 is the degree of the imperfect repair performed in $(0, a_1]$.

Tabulated in Table 1 are the expected numbers of failures $E[N(10)]$ for degrees of repair $\delta_1 \in \{0.1, 0.2, \ldots, 1.0\}$.

Table 1: Expected number of failures in the interval $[0, \tau)$ for various degrees of repair

δ_1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$E[N(10)]$	33.78	27.3	22.4	18.81	16.29	14.64	13.63	13.09	12.86	12.79	12.78

Note that, according to the repair model, as the degree of repair increases, the expected number of failures decreases; also see Figure 4.

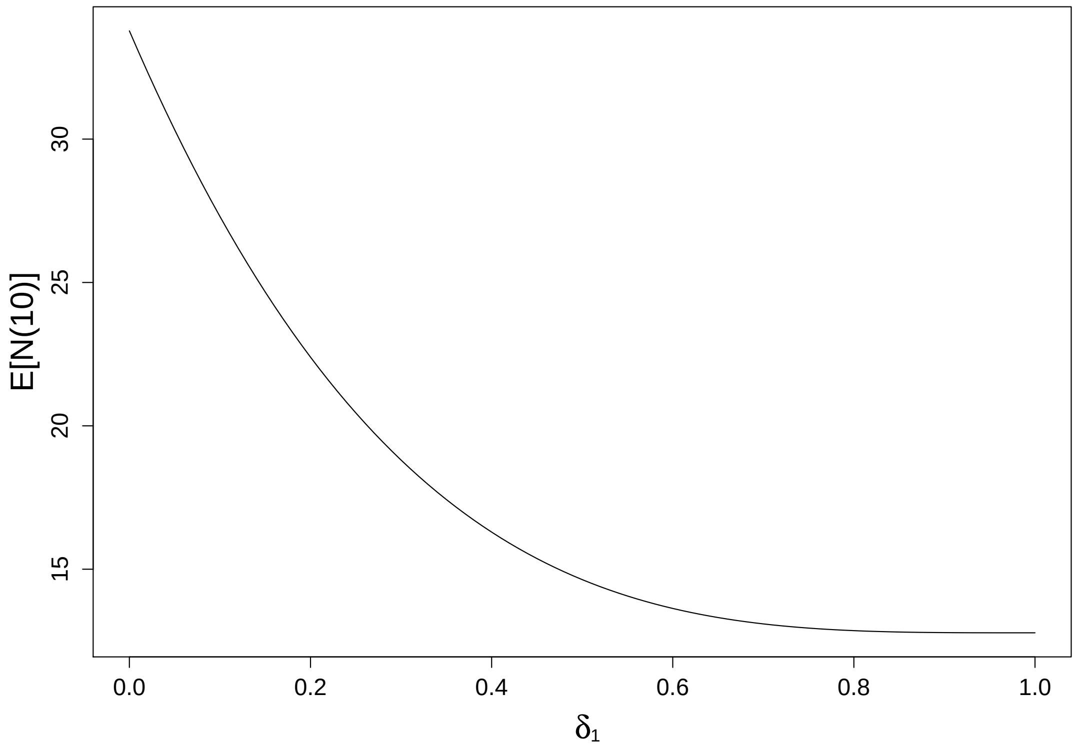

Figure 4: Expected number of failures $E[N(10)]$ for $\delta_1 \in [0, 1]$.

5 Conclusion

In this article, we proposed a new repair model for systems having a BFR function. The effect of repairs was modeled as a modification in the virtual age of the system following the repairs.

According to the proposed model (illustrated in Section 4), as the degree of any given repair increases (while others remain fixed), the expected number of failures decreases, since the reliability of the system is improved.

Some possible future research directions are deriving virtual age models for systems with more than two change points and extension of these models to two dimensions.

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